

Higher order Painlevé system of type $D_{2n+2}^{(1)}$ and monodromy preserving deformation

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Abstract

The higher order Painlevé system of type $D_{2n+2}^{(1)}$ was proposed by Y. Sasano as an extension of P_{VI} for the affine Weyl group symmetry with the aid of algebraic geometry for Okamoto initial value space. In this article, we give it as the monodromy preserving deformation of a Fuchsian system.

Key Words: Painlevé system, Schlesinger system, Laplace transformation.

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1 Introduction

The main object in this article is the higher order Painlevé system of type $D_{2n+2}^{(1)}$ [13]; we call it a *Sasano system*. It is expressed as a Hamiltonian system on $\mathbb{P}^1(\mathbb{C})$

$$\begin{aligned} s(s-1)\frac{dq_i}{ds} &= \frac{\partial H}{\partial p_i}, & s(s-1)\frac{dp_i}{ds} &= -\frac{\partial H}{\partial q_i} \quad (i=1, \dots, n), \\ H &= \sum_{i=1}^n H_i + \sum_{1 \leq i < j \leq n} 2(q_i - s)p_i q_j \{(q_j - 1)p_j + \alpha_{2j}\}, \end{aligned} \tag{1.1}$$

where $H_i = H_i[q_i, p_i; \kappa_{i,s}, \kappa_{i,1}, \kappa_{i,0}, \kappa_{i,\infty}; s]$ is the Hamiltonian for P_{VI} defined by

$$\begin{aligned} H_i &= q_i(q_i - 1)(q_i - s)p_i^2 - (\kappa_{i,s} - 1)q_i(q_i - 1)p_i \\ &\quad - \kappa_{i,1}q_i(q_i - s)p_i - \kappa_{i,0}(q_i - 1)(q_i - s)p_i + \alpha_{2i}(\alpha_{2i} + \kappa_{i,\infty})q_i, \end{aligned}$$

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and

$$\begin{aligned}
\kappa_{i,s} &= \alpha_1 + \sum_{j=1}^{i-1} \alpha_{2j+1}, & \kappa_{i,1} &= \sum_{j=i}^{n-1} \alpha_{2j+1} + \sum_{j=i+1}^n 2\alpha_{2j} + \alpha_{2n+1}, \\
\kappa_{i,0} &= \sum_{j=i}^{n-1} \alpha_{2j+1} + \alpha_{2n+2}, & \kappa_{i,\infty} &= \alpha_0 + \sum_{j=1}^{i-1} 2\alpha_{2j} + \sum_{j=1}^{i-1} \alpha_{2j+1}.
\end{aligned} \tag{1.2}$$

The fixed parameters $\alpha_0, \dots, \alpha_{2n+2}$ satisfy a relation $\alpha_0 + \alpha_1 + \sum_{j=2}^{2n} 2\alpha_j + \alpha_{2n+1} + \alpha_{2n+2} = 1$.

The system (1.1) was proposed as an extension of P_{VI} for the affine Weyl group symmetry with the aid of algebraic geometry for Okamoto initial value space. It was also given as the compatibility condition of the Lax pair associated with a loop algebra $\mathfrak{so}_{4n+4}[z, z^{-1}]$ [1]. But the relationship with the monodromy preserving deformation of a Fuchsian system has not been clarified. The aim of this article is to investigate it.

Recently, higher order generalizations of P_{VI} has been studied from a viewpoint of the monodromy preserving deformations of Fuchsian systems. It is shown in [8, 10] that any irreducible Fuchsian system can be reduced to finite types of systems by using Katz's two operations, addition and middle convolution [7]. It is also shown in [4] that the isomonodromy deformation equation is invariant under Katz's two operations. Those fact allows us to construct a classification theory of the isomonodromy deformation equation.

The Fuchsian systems with two accessory parameters are classified by Kostov [8]. According to it, they are reduced to the systems with the following spectral types:

4 singularities	11, 11, 11, 11
3 singularities	111, 111, 111 22, 1111, 1111 33, 222, 111111

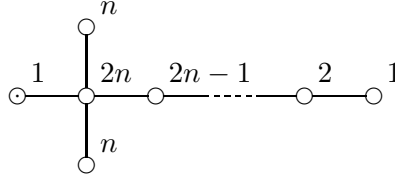
The system with the spectral type $\{11, 11, 11, 11\}$ gives P_{VI} as the monodromy preserving deformation. Note that the other three systems have no deformation parameters.

In general, the Fuchsian systems can be classified with the aid of algorithm proposed by Oshima [10]. The systems with four accessory parameters are reduced as follows:

5 singularities	11, 11, 11, 11, 11
4 singularities	21, 21, 111, 111 31, 22, 22, 1111 22, 22, 22, 211
3 singularities	211, 1111, 1111 221, 221, 11111 32, 11111, 111111
	222, 222, 2211 33, 2211, 111111 44, 2222, 22211
	44, 332, 11111111 55, 3331, 22222 66, 444, 2222211

The system with $\{11, 11, 11, 11, 11\}$ corresponds to the Garnier system in two variables [3]. And the systems with four singularities correspond to four-dimensional Painlevé equations, which are investigated by Sakai [12]. Among them, the system with $\{31, 22, 22, 1111\}$ corresponds to the system (1.1) of the case $n = 2$. In this article, we consider its natural extension. Namely, we consider the Fuchsian system with the spectral type $\{(n, n), (n, n), (2n - 1, 1), (1^{2n})\}$ and show that its monodromy preserving deformation gives the system (1.1).

Remark 1.1. *The choice of a spectral type $\{(n, n), (n, n), (2n - 1, 1), (1^{2n})\}$ is suggested by the recent work of Oshima [10]. According to it, a Fuchsian system with this spectral type corresponds to a Kac-Moody root system with the following Dynkin diagram:*



A dotted circle represents a simple root which is not orthogonal to the other roots.

Remark 1.2. *The Fuchsian system with the spectral type $\{21, 21, 111, 111\}$ corresponds to the fourth order Painlevé system given in [2]. Furthermore the system with the spectral type $\{(n, 1), (n, 1), (1^{n+1}), (1^{n+1})\}$ is systematically investigated by Tsuda. It corresponds to the Schlesinger system $\mathcal{H}_{n+1,1}$ given in [14], or equivalently, the higher order Painlevé system given in [11].*

The other aim of this article is to investigate a relationship between two origins of the Sasano system, the Lax pair associated with $\mathfrak{so}_{4n+4}[z, z^{-1}]$ and the Fuchsian system with the spectral type $\{(n, n), (n, n), (2n - 1, 1), (1^{2n})\}$. It is suggested that those two linear systems are related via a Laplace transformation. In this article, we show it for the case $n = 2$.

This article is organized as follows. In Section 2, we introduce a Fuchsian system with the spectral type $\{(n, n), (n, n), (2n - 1, 1), (1^{2n})\}$ and its monodromy preserving deformation. In Section 3, the system (1.1) is derived from the Schlesinger system given in Section 2. In Section 4, we clarify a relation between two linear systems for the fourth order Sasano system with the aid of a Laplace transformation.

2 Schlesinger system

In this section, following [6, 12], we introduce a Fuchsian system with the spectral type $\{(n, n), (n, n), (2n-1, 1), (1^{2n})\}$ and its monodromy preserving deformation.

Consider a system of linear differential equations on $\mathbb{P}^1(\mathbb{C})$

$$\frac{\partial}{\partial x} Y(x) = \left(\frac{A_t}{x-t} + \frac{A_1}{x-1} + \frac{A_0}{x} \right) Y(x), \quad A_t, A_1, A_0 \in \text{Mat}(2n; \mathbb{C}), \quad (2.1)$$

with regular singularities $x = t, 1, 0, \infty$. Here we assume

1. The data of eigenvalues of residue matrices is given by

$$\begin{array}{cccccc} \theta_t, & \theta_t, & \dots, & \theta_t, & 0, & \dots, & 0 & \text{at } x = t, \\ \theta_1, & \theta_1, & \dots, & \theta_1, & 0, & \dots, & 0 & \text{at } x = 1, \\ \theta_0, & 0, & \dots, & 0, & 0, & \dots, & 0 & \text{at } x = 0, \\ \kappa_1, & \kappa_2, & \dots, & \kappa_n, & \kappa_{n+1}, & \dots, & \kappa_{2n} & \text{at } x = \infty. \end{array}$$

2. Each residue matrix can be diagonalized.

Note that the Fuchsian relation $n\theta_t + n\theta_1 + \theta_0 + \sum_{i=1}^{2n} \kappa_i = 0$ is satisfied. The monodromy preserving deformation of the system (2.1) is described as the Schlesinger system

$$\frac{\partial A_t}{\partial t} = -\frac{[A_t, A_0]}{t} - \frac{[A_t, A_1]}{t-1}, \quad \frac{\partial A_1}{\partial t} = \frac{[A_t, A_1]}{t-1}, \quad \frac{\partial A_0}{\partial t} = \frac{[A_t, A_0]}{t}. \quad (2.2)$$

Note that the residue matrix $A_\infty = -A_t - A_1 - A_0$ at $x = \infty$ is a constant matrix. The system (2.2) can be expressed as a Hamiltonian system

$$\frac{\partial A_\xi}{\partial t} = \{K, A_\xi\} \quad (\xi = t, 1, 0), \quad K = \frac{\text{tr} A_t A_1}{t-1} + \frac{\text{tr} A_t A_0}{t}, \quad (2.3)$$

with the Poisson bracket

$$\{(A_\xi)_{k,l}, (A_{\xi'})_{r,s}\} = \delta_{\xi,\xi'} \{\delta_{r,l}(A_\xi)_{k,s} - \delta_{k,s}(A_\xi)_{r,l}\},$$

where $\delta_{i,j}$ stands for the Kronecker delta.

We consider a gauge transformation $\tilde{A}_\xi = G^{-1} A_\xi G$ ($\xi = t, 1, 0, \infty$) such that

$$\tilde{A}_0 = \begin{bmatrix} \theta_0 & a_2^{(0)} & \dots & a_{2n}^{(0)} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad \tilde{A}_\infty = \begin{bmatrix} \kappa_1 & & & \mathbf{O} \\ a_2^{(\infty)} & \kappa_2 & & \\ \vdots & & \ddots & \\ a_{2n}^{(\infty)} & \mathbf{O} & & \kappa_{2n} \end{bmatrix}.$$

Here the matrix G is decomposed into a product of two matrices as $G = G_1 G_2$, where $G_1^{-1} A_\infty G_1$ is a diagonal matrix and G_2 is a lower triangle matrix of which all entries on the diagonals are one. Then the system (2.3) is transformed into

$$\frac{\partial \tilde{A}_\xi}{\partial t} = \{K, \tilde{A}_\xi\} \quad (\xi = t, 1, 0), \quad K = \frac{\text{tr} \tilde{A}_t \tilde{A}_1}{t-1} + \frac{\text{tr} \tilde{A}_t \tilde{A}_0}{t}, \quad (2.4)$$

with the Poisson bracket

$$\{(\tilde{A}_\xi)_{k,l}, (\tilde{A}_{\xi'})_{r,s}\} = \delta_{\xi,\xi'} \{\delta_{r,l} (\tilde{A}_\xi)_{k,s} - \delta_{k,s} (\tilde{A}_\xi)_{r,l}\}. \quad (2.5)$$

Note that the following relation is satisfied:

$$\tilde{A}_t + \tilde{A}_1 + \tilde{A}_0 + \tilde{A}_\infty = 0. \quad (2.6)$$

In order to derive the canonical Hamiltonian system from (2.4), we use the method established in [5]. Consider a decomposition of matrices A_ξ ($\xi = t, 1$) as

$$\tilde{A}_\xi = \begin{bmatrix} I_n \\ B_\xi \end{bmatrix} [\theta_\xi I_n - C_\xi B_\xi, \quad C_\xi],$$

where $B_\xi = [b_{i,j}^{(\xi)}]$ and $C_\xi = [c_{i,j}^{(\xi)}]$ are $n \times n$ matrices. Then we can regard $b_{i,j}^{(\xi)}$ and $c_{j,i}^{(\xi)}$ as canonical variables. In fact, the Poisson bracket

$$\{b_{i,j}^{(\xi)}, c_{j,i}^{(\xi)}\} = -1 \quad (i, j = 1, \dots, n; \xi = t, 1), \quad \{\text{otherwise}\} = 0. \quad (2.7)$$

implies the one (2.5).

The number of accessory parameters of the system (2.1) is equal to $2n$. Therefore the system (2.4) with (2.6) can be rewritten into the Hamiltonian system of order $2n$, which is just equivalent to (1.1) as we prove below.

3 Sasano system

Under the system (1.1), we define independent and dependent variables by

$$t = 1 - \frac{1}{s}, \quad \lambda_i = 1 - \frac{q_i}{s}, \quad \mu_i = -s p_i \quad (i = 1, \dots, n). \quad (3.1)$$

Then they satisfy a Hamiltonian system

$$\begin{aligned} t(t-1) \frac{d\lambda_i}{dt} &= \frac{\partial H}{\partial \mu_i}, \quad t(t-1) \frac{d\mu_i}{dt} = -\frac{\partial H}{\partial \lambda_i} \quad (i = 1, \dots, n), \\ H &= \sum_{i=1}^n H_i[\lambda_i, \mu_i; \kappa_{i,1}, \kappa_{i,0}, \kappa_{i,s}, \kappa_{i,\infty}; t] \\ &\quad + \sum_{1 \leq i < j \leq n} 2\lambda_i \mu_i (\lambda_j - 1) \{(\lambda_j - t)\mu_j + \alpha_{2j}\}, \end{aligned} \quad (3.2)$$

where $\kappa_{i,s}, \kappa_{i,1}, \kappa_{i,0}, \kappa_{i,\infty}$ are the parameters defined by (1.2). In this section, we derive the system (3.2) from the one (2.4) with (2.6).

Let $\Delta_{j_1, \dots, j_r}^{i_1, \dots, i_r}(A)$ be a minor determinant of A for (i_1, \dots, i_r) -th row and (j_1, \dots, j_r) -th column. Then we arrive at

Theorem 3.1. *Under the system (2.4) with (2.6) and (2.7), we set*

$$\begin{aligned}\mu_i &= (-1)^{n-i} t^{-1} \frac{\Delta_{i,i+1, \dots, n}^{1,i+1, \dots, n}(C_1)}{\Delta_{i+1, \dots, n}^{i+1, \dots, n}(C_t)} \sum_{k=1}^i \frac{\Delta_{k,i+1, \dots, n}^{i,i+1, \dots, n}(C_t)}{\Delta_{i, \dots, n}^{i, \dots, n}(C_t)} b_{k,1}^{(t)}, \\ \lambda_i &= (-1)^{n-i+1} t \frac{\Delta_{i,i+1, \dots, n}^{1,i+1, \dots, n}(C_t)}{\Delta_{i,i+1, \dots, n}^{1,i+1, \dots, n}(C_1)} \quad (i = 1, \dots, n).\end{aligned}\tag{3.3}$$

Then those variables are found out to be canonical coordinates of a $2n$ -dimensional system with the Poisson bracket

$$\{\mu_i, \lambda_j\} = \delta_{i,j}, \quad \{\mu_i, \mu_j\} = \{\lambda_i, \lambda_j\} = 0 \quad (i, j = 1, \dots, n).$$

Furthermore they satisfy the system (3.2) with the parameters

$$\begin{aligned}\alpha_1 &= -\theta_t, \quad \alpha_2 = -\kappa_{n+1}, \quad \alpha_{2i-1} = \theta_t + \theta_1 + \kappa_i + \kappa_{n+i-1}, \\ \alpha_{2i} &= -\theta_t - \theta_1 - \kappa_i - \kappa_{n+i} \quad (i = 2, \dots, n), \\ \alpha_{2n+1} &= \theta_t + \theta_1 + \kappa_1 + \kappa_{2n}, \quad \alpha_{2n+2} = -\kappa_1 + \kappa_{2n} + 1.\end{aligned}\tag{3.4}$$

3.1 Canonical coordinates

In this subsection, we prove the first half of Theorem 3.1.

We can show $\{\mu_i, \lambda_j\} = \delta_{i,j}$ as follows. Denoting $\Delta_{j, \dots, l-1, l+1, \dots, n}^{j+1, \dots, n}$ by $\Delta_{j, \dots, \widehat{l}, \dots, n}^{j+1, \dots, n}$, we have

$$\{\mu_i, \lambda_j\} = \frac{\Delta_{i,i+1, \dots, n}^{1,i+1, \dots, n}(C_1)}{\Delta_{i+1, \dots, n}^{i+1, \dots, n}(C_t)} \sum_{k=1}^i \frac{\Delta_{k,i+1, \dots, n}^{i,i+1, \dots, n}(C_t)}{\Delta_{i, \dots, n}^{i, \dots, n}(C_t)} \sum_{l=j}^n (-1)^{l-j} \frac{\Delta_{j, \dots, \widehat{l}, \dots, n}^{j+1, \dots, n}(C_t)}{\Delta_{j,j+1, \dots, n}^{1,j+1, \dots, n}(C_1)} \delta_{k,l}.\tag{3.5}$$

If $i < j$, the right-hand side of (3.5) turns to be zero. If $i = j$, the right-hand side of (3.5) turns to be one. If $j < i$, then we have

$$(\text{RHS of (3.5)}) = \frac{\Delta_{i,i+1, \dots, n}^{1,i+1, \dots, n}(C_1)}{\Delta_{i+1, \dots, n}^{i+1, \dots, n}(C_t)} \sum_{k=j}^i (-1)^{k-j} \frac{\Delta_{k,i+1, \dots, n}^{i,i+1, \dots, n}(C_t) \Delta_{j, \dots, \widehat{k}, \dots, n}^{j+1, \dots, n}(C_t)}{\Delta_{i, \dots, n}^{i, \dots, n}(C_t) \Delta_{j,j+1, \dots, n}^{1,j+1, \dots, n}(C_1)}.$$

On the other hand, we obtain

$$\begin{aligned}
& \sum_{k=j}^i (-1)^{k-j} \Delta_{k,i+1,\dots,n}^{i,i+1,\dots,n}(C_t) \Delta_{j,\dots,\widehat{k},\dots,n}^{j+1,\dots,n}(C_t) \\
&= \begin{vmatrix} \Delta_{j,i+1,\dots,n}^{i,i+1,\dots,n}(C_t) & \Delta_{j+1,i+1,\dots,n}^{i,i+1,\dots,n}(C_t) & \cdots & \Delta_{i,\dots,n}^{i,\dots,n}(C_t) & 0 & \cdots & 0 \\ c_{j+1,j}^{(t)} & c_{j+1,j+1}^{(t)} & \cdots & c_{j+1,i}^{(t)} & c_{j+1,i+1}^{(t)} & \cdots & c_{j+1,n}^{(t)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n,j}^{(t)} & c_{n,j+1}^{(t)} & \cdots & c_{n,i}^{(t)} & c_{n,i+1}^{(t)} & \cdots & c_{n,n}^{(t)} \end{vmatrix} \\
&= \sum_{l=i}^n (-1)^{l-i} \Delta_{i+1,\dots,n}^{i,\dots,\widehat{l},\dots,n}(C_t) \Delta_{j,j+1,\dots,n}^{l,j+1,\dots,n}(C_t) \\
&= 0.
\end{aligned}$$

Hence the right-hand side of (3.5) turns to be zero.

We can show $\{\mu_i, \mu_j\} = 0$ and $\{\lambda_i, \lambda_j\} = 0$ immediately because rational expressions μ_i and λ_i defined by (3.3) do not contain the canonical variables $c_{1,k}^{(t)}$ and $b_{k,1}^{(t)}$, respectively.

3.2 Derivation of the Sasano system

In this subsection, we prove the second half of Theorem 3.1.

Under the system (2.4) with (2.6), the dependent variables μ_i, λ_i given by (3.3) satisfy

$$\begin{aligned}
\frac{\partial \mu_i}{\partial t} &= \{\tilde{K}, \mu_i\}, \quad \frac{\partial \lambda_i}{\partial t} = \{\tilde{K}, \lambda_i\} \quad (i = 1, \dots, n), \\
\tilde{K} &= \frac{\text{tr} \tilde{A}_t \tilde{A}_1}{t-1} + \frac{\text{tr} \tilde{A}_t \tilde{A}_0}{t} + \sum_{i=1}^n \frac{\mu_i \lambda_i}{t}.
\end{aligned}$$

Hence it is enough to verify that the Hamiltonian \tilde{K} is transformed into the one H given by (3.2) via the transformation (3.3) and (3.4).

First we consider a partition of residue matrix

$$\tilde{A}_\xi = \begin{bmatrix} A_{11}^{(\xi)} & A_{12}^{(\xi)} & A_{13}^{(\xi)} \\ A_{21}^{(\xi)} & A_{22}^{(\xi)} & A_{23}^{(\xi)} \\ A_{31}^{(\xi)} & A_{32}^{(\xi)} & A_{33}^{(\xi)} \end{bmatrix} \quad (\xi = t, 1, 0, \infty),$$

where each block $A_{ij}^{(\xi)}$ is an $n_i \times n_j$ matrix with $(n_1, n_2, n_3) = (1, n-1, n)$. With this block form, the relation (2.6) is described as

$$A_{ij}^{(t)} + A_{ij}^{(1)} + A_{ij}^{(0)} + A_{ij}^{(\infty)} \quad (i, j = 1, 2, 3). \quad (3.6)$$

The Hamiltonian \tilde{K} is given by

$$\begin{aligned}\mathrm{tr}\tilde{A}_t\tilde{A}_1 &= \sum_{i=1}^3 \sum_{j=1}^3 \mathrm{tr}A_{ij}^{(t)}A_{ji}^{(1)}, \\ \mathrm{tr}\tilde{A}_t\tilde{A}_0 &= \theta_0 A_{11}^{(t)} - \mathrm{tr}A_{21}^{(t)}(A_{12}^{(t)} + A_{12}^{(1)}) - \mathrm{tr}A_{31}^{(t)}(A_{13}^{(t)} + A_{13}^{(1)}).\end{aligned}\tag{3.7}$$

Note that $A_{2j}^{(0)} = A_{3j}^{(0)} = 0$ and $A_{12}^{(\infty)} = A_{13}^{(\infty)} = A_{23}^{(\infty)} = A_{32}^{(\infty)} = 0$.

Next we rewrite the Hamiltonian given by (3.7) into the one expressed in terms of the matrices B_t, C_t, B_1, C_1 . Let $E_1 = \mathrm{diag}[1, 0, \dots, 0]$ and $E_{2n} = \mathrm{diag}[0, 1, \dots, 1]$. Then the relation (3.6) implies

$$E_1(C_t B_t + C_1 B_1)E_1 - (\theta_t + \theta_1 + \theta_0 + \kappa_1)E_1 = 0,$$

for $(i, j) = (1, 1)$;

$$E_{2n}(C_t B_t + C_1 B_1)E_{2n} - \mathrm{diag}[0, \theta_t + \theta_1 + \kappa_2, \dots, \theta_t + \theta_1 + \kappa_n] = 0,$$

for $(i, j) = (3, 3)$;

$$E_{2n}(C_t + C_1) = 0,$$

for $(i, j) = (2, 3)$;

$$B_t C_t + B_1 C_1 + \mathrm{diag}[\kappa_{n+1}, \dots, \kappa_{2n}] = 0,$$

for $(i, j) = (3, 3)$. We obtain from them

$$\begin{aligned}\mathrm{tr}\tilde{A}_t\tilde{A}_1 &= (\mathrm{tr}E_1 C_t B_t)(\mathrm{tr}E_1 C_1 B_1) - (\mathrm{tr}E_1 C_t B_t)(\mathrm{tr}E_1 C_1 B_t) - 2(\mathrm{tr}E_1 C_t B_t)^2 \\ &\quad - \mathrm{tr}E_1 C_1 (B_t - B_1)E_{2n}C_t B_t - \mathrm{tr}E_1 C_t (B_t - B_1)E_{2n}C_1 B_1 \\ &\quad + (3\theta_t + \theta_1 + 2\theta_0 + 2\kappa_1)\mathrm{tr}E_1 C_t B_t - (\theta_t + \theta_0 + \kappa_1)\mathrm{tr}E_1 C_t B_1 \\ &\quad + \theta_t \mathrm{tr}E_1 C_1 B_t + n\theta_t \theta_1 - \frac{1}{2} \sum_{i=2}^n (\theta_t + \theta_1 - \kappa_i)(\theta_t + \theta_1 + \kappa_i) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \kappa_{n+i}^2 - \frac{1}{2}(\theta_t + \theta_1 + \theta_0 + \kappa_1)^2 - \theta^t(\theta_t + \theta_1 + \theta_0 + \kappa_1),\end{aligned}$$

and

$$\begin{aligned}\mathrm{tr}\tilde{A}_t\tilde{A}_0 &= (\mathrm{tr}E_1 C_t B_t)(\mathrm{tr}E_1 C_1 B_t) + (\mathrm{tr}E_1 C_t B_t)^2 + \mathrm{tr}E_1 C_1 (B_t - B_1)E_{2n}C_t B_t \\ &\quad - (\theta_t + \theta_0)\mathrm{tr}E_1 C_t B_t - \theta_t \mathrm{tr}E_1 C_1 B_t + \theta_t \theta_0.\end{aligned}$$

In order to derive the Hamiltonian H given by (3.2), we introduce the following lemma.

Lemma 3.2. *We have relations*

$$\begin{aligned}
\operatorname{tr} E_1 C_t B_t &= - \sum_{i=1}^n \lambda_i \mu_i, \quad \operatorname{tr} E_1 C_1 B_t = t \sum_{i=1}^n \mu_i, \\
\operatorname{tr} E_1 C_t B_1 &= - \frac{1}{t} \sum_{i=1}^n \lambda_i (\lambda_i \mu_i + \beta_i), \\
\operatorname{tr} E_1 C_1 (B_t - B_1) E_{2n} C_t B_t \\
&= t \sum_{i=1}^n \mu_i \left\{ - \sum_{j=1}^{i-1} (\lambda_j \mu_j + \beta_j) - \beta_i - \kappa_{n+i} + \sum_{j=i+1}^n \lambda_j \mu_j \right\}, \\
\operatorname{tr} E_1 C_t (B_t - B_1) E_{2n} C_t B_1 \\
&= - \frac{1}{t} \sum_{i=1}^n \lambda_i (\lambda_i \mu_i + \beta_i) \left\{ \sum_{j=1}^{i-1} \lambda_j \mu_j - \beta_i - \kappa_{n+i} - \sum_{j=i+1}^n (\lambda_j \mu_j + \beta_j) \right\},
\end{aligned}$$

where

$$\beta_1 = -\kappa_{n+1}, \quad \beta_i = -\theta_t - \theta_1 - \kappa_i - \kappa_{n+i} \quad (i = 2, \dots, n).$$

Proof. We only prove the first relation here. The other ones can be proved in a similar way.

We take an $n \times n$ matrix

$$P = \begin{bmatrix} f_{1,1} & & & O \\ f_{2,1} & f_{2,2} & & \\ \vdots & \vdots & \ddots & \\ f_{n,1} & f_{n,2} & \cdots & f_{n,n} \end{bmatrix}, \quad f_{i,k} = (-1)^{n-i} t^{-1} \frac{\Delta_{i,i+1,\dots,n}^{1,i+1,\dots,n}(C_1) \Delta_{k,i+1,\dots,n}^{i,i+1,\dots,n}(C_t)}{\Delta_{i+1,\dots,n}^{i+1,\dots,n}(C_t) \Delta_{i,\dots,n}^{i,\dots,n}(C_t)}.$$

Its inverse matrix of is given by

$$P^{-1} = \begin{bmatrix} g_{1,1} & & & O \\ g_{2,1} & g_{2,2} & & \\ \vdots & \vdots & \ddots & \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{bmatrix}, \quad g_{k,i} = (-1)^{n-k} t \frac{\Delta_{i,\dots,\widehat{k},\dots,n}^{i+1,\dots,n}(C_t)}{\Delta_{i,i+1,\dots,n}^{1,i+1,\dots,n}(C_1)}.$$

Note that we derive an explicit formula of P^{-1} by using the Plücker relations for matrices. Then we can rewrite (3.3) into

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = P \begin{bmatrix} b_{1,1}^{(t)} \\ \vdots \\ b_{n,1}^{(t)} \end{bmatrix}, \quad [\lambda_1 \quad \cdots \quad \lambda_n] = - \begin{bmatrix} c_{1,1}^{(t)} & \cdots & c_{n,1}^{(t)} \end{bmatrix} P^{-1}. \quad (3.8)$$

On the other hand, an adjoint action of P implies

$$\mathrm{tr} E_1 C_t B_t = \mathrm{tr} P \begin{bmatrix} b_{1,1}^{(t)} \\ \vdots \\ b_{n,1}^{(t)} \end{bmatrix} \begin{bmatrix} c_{1,1}^{(t)} & \dots & c_{n,1}^{(t)} \end{bmatrix} P^{-1}. \quad (3.9)$$

Combining (3.8) and (3.9), we derive the first relation. \square

4 Laplace transformation

As is seen in the previous section, the system (1.1) is derived from the Fuchsian system. On the other hand, in the previous work [1], it was also derived from the Lax pair associated with the loop algebra $\mathfrak{so}_{4n+4}[z, z^{-1}]$. In this section, we clarify a relation between those two linear systems with the aid of a Laplace transformation for the case $n = 2$.

We recall the definition of the loop algebra \mathfrak{so}_{2N} for $N \geq 3$. Let $E_{i,j}$ be a $2N \times 2N$ matrix with 1 on the (i, j) -th entry and zeros elsewhere. We also set $X_{i,j} = E_{i,j} - E_{2N+1-j, 2N+1-i}$. Then the loop algebra $\mathfrak{so}_{2N}[z, z^{-1}]$ is generated by

$$\begin{aligned} e_0 &= zX_{2N-1,1}, & e_i &= X_{i,i+1} \quad (i = 1, \dots, N-1), & e_N &= X_{N-1,N+1}, \\ f_0 &= \frac{1}{z}X_{1,2N-1}, & f_i &= X_{i+1,i} \quad (i = 1, \dots, N-1), & f_N &= X_{N+1,N-1}, \end{aligned}$$

and $h_i = X_{i,i}$ ($i = 1, \dots, N$). In the following, we use a notation

$$e_{i_1, \dots, i_{n-1}, i_n} = \mathrm{ade}_{i_1} \dots \mathrm{ade}_{i_{n-1}}(e_{i_n}), \quad \mathrm{ade}_i(e_j) = [e_i, e_j].$$

Note that the algebra \mathfrak{so}_{2N} is defined by

$$\mathfrak{so}_{2N} = \left\{ X \in \mathrm{Mat}(2n; \mathbb{C}) \mid JX + {}^t X J = 0 \right\}, \quad J = \sum_{i=1}^{2N} E_{i, 2N+1-i}.$$

The system (1.1) of the case $n = 2$ is given as the compatibility condition of the Lax pair

$$z \frac{\partial}{\partial z} \Psi_{12}(z) = M_{12}(z) \Psi_{12}(z), \quad \frac{\partial}{\partial s} \Psi_{12}(z) = B_{12}(z) \Psi_{12}(z). \quad (4.1)$$

The matrix $M_{12}(z) \in \mathfrak{so}_{12}[z, z^{-1}]$ is described as

$$M_{12}(z) = \sum_{i=1}^6 \varepsilon_i h_i - e_0 + \sum_{i=1}^6 \varphi_i e_i - e_{1,2} - e_{2,3} - e_{3,4} - e_{4,5} - e_{4,6},$$

where

$$\begin{aligned}\varepsilon_1 &= \frac{1}{2}(-1 + \alpha_0 - \alpha_1), & \varepsilon_2 &= \frac{1}{2}(-1 + \alpha_0 + \alpha_1), & \varepsilon_3 &= \frac{1}{2}(-1 + \alpha_0 + \alpha_1 + 2\alpha_2), \\ \varepsilon_4 &= \frac{1}{2}(-2\alpha_4 - \alpha_5 - \alpha_6), & \varepsilon_5 &= \frac{1}{2}(-\alpha_5 - \alpha_6), & \varepsilon_6 &= \frac{1}{2}(\alpha_5 - \alpha_6),\end{aligned}$$

and

$$\varphi_1 = s - q_1, \quad \varphi_2 = p_1, \quad \varphi_3 = q_1 - q_2, \quad \varphi_4 = p_2, \quad \varphi_5 = q_2 - 1, \quad \varphi_6 = q_2.$$

The matrix $B_{12}(z) \in \mathfrak{so}_{12}[z, z^{-1}]$ is described as

$$B_{12}(z) = \sum_{i=1}^6 u_i h_i + \sum_{i=0}^6 v_i e_i + e_{0,2} + v_7 e_{2,3} + v_8 e_{3,4} + v_9 e_{4,5} + v_{10} e_{4,6} + v_{11} e_{2,3,4},$$

where the coefficients u_i, v_i are polynomials in (q_1, q_2, p_1, p_2) ; we do not give their explicit formulas here. In this section, we reduce it to a Fuchsian system with a spectral type $\{31, 22, 22, 1111\}$.

4.1 From $\mathfrak{so}_{12}[z, z^{-1}]$ to $\mathfrak{so}_{10}[z, z^{-1}]$

Under the system (4.1), we consider a gauge transformation

$$\Psi_{12}(z) \rightarrow \tau_{12}(z) \Psi_{12}^*(z),$$

where a function $\tau_{12}(z)$ satisfies

$$z \frac{\partial}{\partial z} \log \tau_{12}(z) = \varepsilon_1 + 1, \quad \frac{\partial}{\partial s} \log \tau_{12}(z) = u_1.$$

We also consider a Laplace transformation

$$\frac{\partial}{\partial z} \Psi_{12}^*(z) \rightarrow \zeta \Phi_{12}(\zeta^{-1}), \quad z \Psi_{12}^*(z) \rightarrow -\frac{\partial}{\partial \zeta} \Phi_{12}(\zeta^{-1}),$$

and a Möbius transformation $\zeta \rightarrow z^{-1}$. Then we obtain

$$z \frac{\partial}{\partial z} \Phi_{12}(z) = N_{12}(z) \Phi_{12}(z), \quad \frac{\partial}{\partial s} \Phi_{12}(z) = C_{12}(z) \Phi_{12}(z), \quad (4.2)$$

with

$$\begin{aligned}N_{12}(z) &= (I_{12} - zM_{12,1})^{-1}(M_{12,0} - \varepsilon_1 I_{12}), \\ C_{12}(z) &= B_{12,0} - u_1 I_{12} + zB_{12,1}(I_{12} - zM_{12,1})^{-1}(M_{12,0} - \varepsilon_1 I_{12}),\end{aligned}$$

where $M_{12}(z) = M_{12,0} + zM_{12,1}$ and $B_{12}(z) = B_{12,0} + zB_{12,1}$. Note that $(M_{12,1})^2 = O$, namely, $(I_{12} - zM_{12,1})^{-1} = I_{12} + zM_{12,1}$. The first columns of $N_{12}(z)$ and $C_{12}(z)$ are both equivalent to the zero vectors. Hence we can reduce the system (4.2) to the one with 11×11 matrices

$$z \frac{\partial}{\partial z} \Psi_{11}(z) = M_{11}(z) \Psi_{11}(z), \quad \frac{\partial}{\partial s} \Psi_{11}(z) = B_{11}(z) \Psi_{11}(z),$$

or equivalently

$$z \frac{\partial}{\partial z} \Psi_{11}^{-1}(z) = -\Psi_{11}^{-1}(z) M_{11}(z), \quad \frac{\partial}{\partial s} \Psi_{11}^{-1}(z) = -\Psi_{11}^{-1}(z) B_{11}(z). \quad (4.3)$$

Under the system (4.3), we consider a gauge transformation

$$\Psi_{11}^{-1}(z) \rightarrow \tau_{11}(z) \Psi_{11}^*(z), \quad z \frac{\partial}{\partial z} \log \tau_{11}(z) = -2\varepsilon_1 - 1, \quad \frac{\partial}{\partial s} \log \tau_{11}(z) = -2u_1,$$

a Laplace transformation

$$\frac{\partial}{\partial z} \Psi_{11}^*(z) \rightarrow \zeta \Phi_{11}(\zeta^{-1}), \quad z \Psi_{11}^*(z) \rightarrow -\frac{\partial}{\partial \zeta} \Phi_{11}(\zeta^{-1}),$$

and a Möbius transformation $\zeta \rightarrow z^{-1}$. Then we obtain

$$z \frac{\partial}{\partial z} \Phi_{11}(z) = -\Phi_{11}(z) N_{11}(z), \quad \frac{\partial}{\partial s} \Phi_{11}(z) = -\Phi_{11}(z) C_{11}(z), \quad (4.4)$$

with

$$N_{11}(z) = (M_{11,0} + 2\varepsilon_1 I_{11})(I_{11} + zM_{11,1})^{-1}, \\ C_{11}(z) = B_{11,0} + 2u_1 I_{11} + z(M_{11,0} + 2\varepsilon_1 I_{11})(I_{11} + zM_{11,1})^{-1} B_{11,1},$$

where $M_{11}(z) = M_{11,0} + zM_{11,1}$ and $B_{11}(z) = B_{11,0} + zB_{11,1}$. Note that $(M_{11,1})^2 = O$, namely, $(I_{11} + zM_{11,1})^{-1} = I_{11} - zM_{11,1}$. The 11-th rows of $N_{11}(z)$ and $C_{11}(z)$ are both equivalent to the zero vectors. Hence we can reduce the system (4.4) to the one associated with $\mathfrak{so}_{10}[z, z^{-1}]$.

Furthermore, we consider a Dynkin diagram automorphism

$$e_i \rightarrow e_{5-i}, \quad h_i \rightarrow h_{5-i} \quad (i = 0, \dots, 5), \quad z \frac{\partial}{\partial z} \rightarrow z \frac{\partial}{\partial z} + \frac{1}{2} \sum_{i=1}^5 h_i.$$

We finally obtain

$$z \frac{\partial}{\partial z} \Psi_{10}(z) = M_{10}(z) \Psi_{10}(z), \quad \frac{\partial}{\partial s} \Psi_{10}(z) = B_{10}(z) \Psi_{10}(z). \quad (4.5)$$

The matrix $M_{10}(z) \in \mathfrak{so}_{10}[z, z^{-1}]$ is described as

$$\begin{aligned} M_{10}(z) = & \sum_{i=1}^5 (-\varepsilon_{7-i} - \frac{1}{2})h_i + \sum_{i=0}^4 \varphi_{6-i}e_i + \{(q_i - s)p_1 - \alpha_1\}e_5 \\ & + e_{0,2} + e_{1,2} + e_{2,3} + e_{3,4} + (q_1 - s)e_{3,5} - e_{5,3,4}. \end{aligned}$$

The matrix $B_{10}(z) \in \mathfrak{so}_{10}[z, z^{-1}]$ is described as

$$\begin{aligned} B_{10}(z) = & \sum_{i=1}^5 u'_i h_i + \sum_{i=0}^5 v'_i e_i + v'_6 e_{0,2} + v'_7 e_{1,2} + v'_8 e_{2,3} + v'_9 e_{3,4} \\ & + v'_{10} e_{3,5} + v'_{11} e_{2,3,4} + v'_{12} e_{2,3,5} + v'_{13} e_{5,3,4} + v'_{14} e_{5,2,3,4}, \end{aligned}$$

where the coefficients u'_i, v'_i are polynomials in (q_1, q_2, p_1, p_2) ; we do not give their explicit formulas here.

4.2 From $\mathfrak{so}_{10}[z, z^{-1}]$ to $\mathfrak{so}_8[z, z^{-1}]$

Similarly as in the previous section, the system (4.5) can be reduced to the one associated with $\mathfrak{so}_8[z, z^{-1}]$. Furthermore, we consider a gauge transformation

$$\Psi_8(z) \rightarrow \exp(-\frac{1}{\varphi_5}e_1) \exp(-e_4) \exp(h_1 \log \varphi_5) \Psi_8(z),$$

the Bäcklund transformation for the Sasano system

$$p_2 \rightarrow p_2 + \frac{\alpha_5}{1 - q_2}, \quad \alpha_4 \rightarrow \alpha_4 + \alpha_5, \quad \alpha_5 \rightarrow -\alpha_5, \quad \alpha_0 \rightarrow \alpha_0 + \alpha_5,$$

and a Dynkin diagram automorphism $e_1 \leftrightarrow e_4, h_1 \leftrightarrow h_4$. We finally obtain

$$z \frac{\partial}{\partial z} \Psi_8(z) = M_8(z) \Psi_8(z), \quad \frac{\partial}{\partial s} \Psi_8(z) = B_8(z) \Psi_8(z). \quad (4.6)$$

The matrix $M_8(z) \in \mathfrak{so}_8[z, z^{-1}]$ is described as

$$\begin{aligned} M_8(z) = & \sum_{i=1}^4 \varepsilon''_{i+1} h_i + \sum_{i=0}^4 \varphi''_i e_i + q_1 e_{0,2} + (s - q_2) e_{1,2} + e_{2,3} \\ & + (1 - q_1) e_{2,4} + s e_{0,1,2} + e_{0,2,3} + (1 - s) e_{1,2,4} + e_{3,2,4}, \end{aligned}$$

where

$$\begin{aligned} \varepsilon''_1 &= \frac{1}{2}(\alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 + 2\alpha_6 - 2), \quad \varepsilon''_2 = \frac{1}{2}(-\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5), \\ \varepsilon''_3 &= \frac{1}{2}(-\alpha_2 - \alpha_4 - \alpha_5), \quad \varepsilon''_4 = \frac{1}{2}(\alpha_2 - \alpha_4 - \alpha_5), \end{aligned}$$

and

$$\begin{aligned}\varphi_0'' &= q_2 p_2 + \alpha_4, & \varphi_1'' &= (q_1 - s)p_1 + \alpha_0 + \alpha_2, \\ \varphi_2'' &= q_1 - q_2, & \varphi_3'' &= p_1, & \varphi_4'' &= (q_2 - 1)p_2 + \alpha_4.\end{aligned}$$

The matrix $B_8(z) \in \mathfrak{so}_8[z, z^{-1}]$ is described as

$$\begin{aligned}B_8(z) &= \sum_{i=1}^4 u_i'' h_i + \sum_{i=0}^4 v_i'' e_i + v_5'' e_{0,2} + v_6'' e_{1,2} + v_7'' e_{2,3} \\ &\quad + v_8'' e_{2,4} + v_9'' e_{0,2,1} + v_{10}'' e_{0,2,3} + v_{11}'' e_{1,2,4} + v_{12}'' e_{3,2,4},\end{aligned}$$

where the coefficients u_i'', v_i'' are polynomials in (q_1, q_2, p_1, p_2) ; we do not give their explicit formulas here.

4.3 From $\mathfrak{so}_8[z, z^{-1}]$ to \mathfrak{sl}_4

Similarly as in the previous section, the system (4.6) can be reduced to a Fuchsian system

$$z \frac{\partial}{\partial z} \Psi_7(z) = M_7(z) \Psi_7(z), \quad \frac{\partial}{\partial s} \Psi_7(z) = B_7(z) \Psi_7(z), \quad (4.7)$$

with 7×7 matrices $M_7(z) = M_{7,0} + zM_{7,1}$ and $B_7(z) = B_{7,0} + zB_{7,1}$. Then we have $\det(I_7 + zM_{7,1}) = (z - s)(z - s + 1)$. It follows that the system (4.7) can be reduced to a Fuchsian one with 6×6 matrices. It is described as

$$z \frac{\partial}{\partial z} \Psi_6(z) = M_6(z) \Psi_6(z), \quad \frac{\partial}{\partial s} \Psi_6(z) = B_6(z) \Psi_6(z), \quad (4.8)$$

with

$$M_6(z) = \frac{M_{6,0} + zM_{6,1} + z^2M_{6,2}}{(z - s)(z - s + 1)}, \quad B_6(z) = \frac{B_{6,0} + zB_{6,1} + z^2B_{6,2}}{(z - s)(z - s + 1)},$$

where $M_{6,i}, B_{6,i} \in \mathfrak{so}_6$ ($i = 0, 1, 2$); we do not give their explicit formulas here.

Recall that the algebra \mathfrak{so}_6 is isomorphic to the one \mathfrak{sl}_4 . With the aid of this fact, we can reduce the system (4.8) to a Fuchsian system of fourth order. Furthermore, we consider a transformation of independent and dependent variables

$$x = \frac{z}{s}, \quad t = 1 - \frac{1}{s}, \quad \lambda_i = 1 - \frac{q_i}{s}, \quad \mu_i = -sp_i \quad (i = 1, 2).$$

Note that it arises from the transformation (3.1) of the case $n = 2$. We finally obtain

$$\frac{\partial}{\partial x}\Psi_4(x) = M_4(x)\Psi_4(x), \quad \frac{\partial}{\partial t}\Psi_4(x) = B_4(x)\Psi_4(x), \quad (4.9)$$

with

$$M_4(x) = \frac{M_{4,t}}{x-t} + \frac{M_{4,1}}{x-1} + \frac{M_{4,0}}{x}, \quad B_4(x) = -\frac{M_{4,t}}{x-t} + B_{4,\infty}.$$

Its compatibility condition implies the system (3.2). We do not give explicit formulas of residue matrices here.

By a direct computation, we arrive at

Theorem 4.1. *The system (4.9) is a Fuchsian system with a spectral type $\{31, 22, 22, 1111\}$.*

Remark 4.2. *The system (4.9) can be transformed into the one (2.1) of the case $n = 2$ via certain gauge transformation and Bäcklund transformation.*

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